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These two values of r' give

$$r_1' = c\,\mu + \sqrt{a^2 - c^2 + c^2\,\mu^2},\quad r'_2 = c\,\mu - \sqrt{a^2 - c^2 + c^2\,\mu^2}.$$

$$r'^{\,2}_1 - r'^{\,2}_2 = 4\,c\,\mu\sqrt{a^2 - c^2 + c^2\,\mu^2}$$

$$r'^{\,2}_1 = 2\,c^2\,\mu^2 + a^2 - c^2 + 2\,c\,\mu\sqrt{a^2 - c^2 + c^2\,\mu^2}.$$

The limits of μ for an external point are $\mu = +1$, and $c^2\,\mu^2 = c^2 - a^2$; and for an internal point the limits of μ are $+1$ and -1. The equation (14) now gives, if $c^2\,\mu'^2 = c^2 - a^2$,

$$V = 4\,\pi\,\rho\int_{+\mu'}^{+1} c\,\mu\,d\,\mu\sqrt{a^2 - c^2 + c^2\,\mu^2} = \frac{4\,\pi\,\rho\,a^3}{3\,c} = \frac{\text{mass}}{c}.$$

Equation (15) gives

$$V = 2\,\pi\,\rho\int_{-1}^{+1} d\mu\,(2\,c^2\,\mu^2 + a^2 - c^2 + 2\,c\,\mu\sqrt{a^2 - c^2 + c^2\,\mu^2}) = 2\,\pi\,\rho\,a^2 - \tfrac{2}{3}\,\pi\,\rho\,c^2.$$

We can find the potential of a homogeneous spheroid of revolution, for a point situated in the axis of revolution, by a similar process, though the work is more complicated. Since the values of X, Y, Z can be calculated directly, we can find V for the spheroid by equation (5), the value of C being made the value of V for the center of the spheroid.

SOLUTION OF TWO SIMILAR INDETERMINATE PROBLEMS.

BY GEORGE R. PERKINS, LL. D., UTICA, N. Y.

1. *Find three square numbers in arithmetical progression, such that if from each its root be subtracted, the remainders shall be squares.*

2. *Find three square numbers in arithmetical progression, such that if to each its root be added, the sums shall be squares.*

We will assume for the three square numbers as follows:

$$\left\{\tfrac{1}{4}(x+x^{-1})\pm\tfrac{1}{2}\right\}^2;\quad \left\{\tfrac{1}{4}(y+y^{-1})\pm\tfrac{1}{2}\right\}^2;\quad \left\{\tfrac{1}{4}(z+z^{-1})\pm\tfrac{1}{2}\right\}^2.$$

The *upper* sign corresponds with the first problem, and the *lower* sign with the second problem.

Since the following condition is true for all values of x,

$$\left\{\tfrac{1}{4}(x+x^{-1})\pm\tfrac{1}{2}\right\}^2 \mp \left\{\tfrac{1}{4}(x+x^{-1})\pm\tfrac{1}{2}\right\} = \left\{\tfrac{1}{4}(x-x^{-1})\right\}^2, \ldots\ldots(1)$$

we see that the square numbers as above assumed, will satisfy the conditions of the problems as to the square of the remainders in the *first*, and the square of the sums in the *second.*

It only remains therefore to determine x, y and z so that these assumed numbers shall be in arithmetical progression. That is, we must have

$$\left.\begin{array}{l}\left\{\tfrac{1}{4}(y+y^{-1})\pm\tfrac{1}{2}\right\}^2-\left\{\tfrac{1}{4}(x+x^{-1})\pm\tfrac{1}{2}\right\}^2=\\ \left\{\tfrac{1}{4}(z+z^{-1})\pm\tfrac{1}{2}\right\}^2-\left\{\tfrac{1}{4}(y+y^{-1})\pm\tfrac{1}{2}\right\}^2\end{array}\right\}\cdots\cdots\cdots\cdots(2)$$

Since the difference of the squares of two quantities is equal to the product of their sum into their difference, we must have

$$\left.\begin{array}{l}\left\{\tfrac{1}{4}(y+y^{-1})+\tfrac{1}{4}(x+x^{-1})\pm1\right\}\times\left\{\tfrac{1}{4}(y+y^{-1})-\tfrac{1}{4}(x+x^{-1})\right\}=\\ \left\{\tfrac{1}{4}(z+z^{-1})+\tfrac{1}{4}(y+y^{-1})\pm1\right\}\times\left\{\tfrac{1}{4}(z+z^{-1})-\tfrac{1}{4}(y+y^{-1})\right\}\end{array}\right\}\cdots(3)$$

Condition (3) will be satisfied by the following assumed values:

$$\tfrac{1}{4}(y+y^{-1})+\tfrac{1}{4}(x+x^{-1})\pm1=m\cdots\cdots\cdots\cdots\cdots\cdots(4)$$

$$\tfrac{1}{4}(y+y^{-1})-\tfrac{1}{4}(x+x^{-1})=n\cdots\cdots\cdots\cdots\cdots\cdots(5)$$

$$\tfrac{1}{4}(z+z^{-1})+\tfrac{1}{4}(y+y^{-1})\pm1=\frac{(p+1)m}{p}\cdots\cdots\cdots\cdots(6)$$

$$\tfrac{1}{4}(z+z^{-1})-\tfrac{1}{4}(y+y^{-1})=\frac{p\,n}{p+1}\cdots\cdots\cdots\cdots(7)$$

Conditions (4) and (5) give

$$\tfrac{1}{2}(y+y^{-1})\pm1=m+n\cdots\cdots\cdots\cdots\cdots\cdots(8)$$

$$\tfrac{1}{2}(x+x^{-1})\pm1=m-n\cdots\cdots\cdots\cdots\cdots\cdots(9)$$

Conditions (6) and (7) give

$$\tfrac{1}{2}(z+z^{-1})\pm1=\frac{(p+1)\,m}{p}+\frac{p\,n}{p+1}\cdots\cdots\cdots\cdots(10)$$

$$\tfrac{1}{2}(y+y^{-1})\pm1=\frac{(p+1)\,m}{p}-\frac{p\,n}{p+1}\cdots\cdots\cdots\cdots(11)$$

Putting the values of $\tfrac{1}{2}(y+y^{-1})\pm1$, given by (8) and (11), equal, we find

$$n=\frac{(p+1)\,m}{p(2\,p+1)}\cdots\cdots\cdots\cdots\cdots\cdots(12)$$

Hence, we have

$$\tfrac{1}{2}(x+x^{-1})\pm1=\frac{(2\,p^2-1)\,m}{p(2\,p+1)}\cdots\cdots\cdots\cdots(13)$$

$$\tfrac{1}{2}(y+y^{-1})\pm1=\frac{(2\,p^2+2\,p+1)\,m}{p(2\,p+1)}\cdots\cdots\cdots\cdots(14)$$

$$\tfrac{1}{2}(z + z^{-1}) \pm 1 = \frac{(2p^2 + 4p + 1)\,m}{p(2p + 1)} \ldots\ldots\ldots\ldots (15)$$

Putting $a = \dfrac{2p(2p+1)}{2p^2 - 1}$; $b = \dfrac{2p(2p+1)}{2p^2 + 2p + 1}$; $c = \dfrac{2p(2p+1)}{2p^2 + 4p + 1}$,

and solving (13), (14) and (15), we obtain

$$x = \mp 1 + \frac{2}{a}\left\{ m + \sqrt{m^2 \mp a\,m} \right\} \ldots\ldots\ldots\ldots (16)$$

$$y = \mp 1 + \frac{2}{b}\left\{ m + \sqrt{m^2 \mp b\,m} \right\} \ldots\ldots\ldots\ldots (17)$$

$$z = \mp 1 + \frac{2}{c}\left\{ m + \sqrt{m^2 \mp c\,m} \right\} \ldots\ldots\ldots\ldots (18)$$

We must now determine m so as to make these values of x, y and z rational, that is, we must have

$$m^2 \mp a\,m = \square;\ m^2 \mp b\,m = \square;\ m^2 \mp c\,m = \square \ldots\ldots (a)$$

If we assume $m^2 + a\,m = (m + k)^2$, we shall find

$$m = \frac{k^3}{\pm(2k - a)} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (19)$$

This causes the other conditions of (a) to become

$$\left.\begin{array}{l} m^2 \mp b\,m = \dfrac{k^2\{k^2 - 2bk + ab\}}{(2k - a)^2} \\[2ex] m^2 \mp c\,m = \dfrac{k^2\{k^2 - 2ck + ac\}}{(2k - a)^2} \end{array}\right\} \ldots\ldots\ldots\ldots\ldots (b)$$

Hence, we must have

$$\left.\begin{array}{l} k^2 - 2bk + ab = \square \\ k^2 - 2ck + ac = \square \end{array}\right\} \ldots\ldots\ldots\ldots\ldots\ldots (c)$$

Assume, $k^2 - 2bk + ab = (k - l)^2$, and we obtain

$$k = \frac{l^2 - ab}{2(l - b)} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (20)$$

And the second condition (c) becomes

$$k^2 - 2ck + ac = \frac{l^4 - 4cl^3 + 2(2ac + 2bc - ab)l^2 - 4abcl + a^2b^2}{4(l - b)^2}.$$

Consequently,

$$l^4 - 4cl^3 + 2(2ac + 2bc - ab)l^2 - 4abcl + a^2b^2 = \square \ldots (21)$$

Assuming as follows:

$$l^4 - 4cl^3 + 2(2ac + 2bc - ab)l^2 - 4abcl + a^2b^2$$

$$=\{ l^2 - 2\,c\,l + 2\,c(a + b - c) - a\,b \}^2, \text{ we find}$$

$$l = \tfrac{1}{2}(a + b - c) \dots\dots\dots\dots\dots\dots\dots (22)$$

This value substituted in (20) gives

$$k = \frac{2(a\,b + a\,c + b\,c) - (a^2 + b^2 + c^2)}{4(- a + b + c)} \dots\dots\dots (23)$$

And substituting this value of k in (19) we find

$$m = \frac{\{ 2(a\,b + a\,c + b\,c) - (a^2 + b^2 + c^2 \}^2}{\pm 8(- a + b + c) \,.\, (a - b + c) \,.\, (a + b - c)} \dots (24)$$

Conditions (13), (14) and (15) give

$$\tfrac{1}{4}(x + x^{-1}) \pm \tfrac{1}{2} = \frac{m}{a} \dots\dots\dots\dots\dots\dots (25)$$

$$\tfrac{1}{4}(y + y^{-1}) \pm \tfrac{1}{2} = \frac{m}{b} \dots\dots\dots\dots\dots\dots (26)$$

$$\tfrac{1}{4}(z + z^{-1}) \pm \tfrac{1}{2} = \frac{m}{c} \dots\dots\dots\dots\dots\dots (27)$$

Hence, the three square numbers sought are

$$\left(\frac{m}{a}\right)^2; \quad \left(\frac{m}{b}\right)^2; \quad \left(\frac{m}{c}\right)^2.$$

These roots in terms of p are as follows:

$$\frac{m}{a} = \frac{(2\,p^2 - 1)A^2}{\pm 8(2\,p^2 - 1) \,.\, (2\,p^2 + 2\,p + 1) \,.\, (2\,p^2 + 4\,p + 1)\,B\,C\,D} \dots (31)$$

$$\frac{m}{b} = \frac{(2\,p^2 + 2\,p + 1)A^2}{\pm 8(2\,p^2 - 1) \,.\, (2\,p^2 + 2\,p + 1) \,.\, (2\,p^2 + 4\,p + 1)\,B\,C\,D} \dots (32)$$

$$\frac{m}{c} = \frac{(2\,p^2 + 4\,p + 1)A^2}{\pm 8(2\,p^2 - 1) \,.\, (2\,p^2 + 2\,p + 1) \,.\, (2\,p^2 + 4\,p + 1)\,B\,C\,D} \dots (33)$$

In the above, we have

$A = 48\,p^8 + 192\,p^7 + 224\,p^6 - 280\,p^4 - 336\,p^3 - 184\,p^2 - 48\,p - 5.$

$B = 4\,p^4 - 12\,p^2 - 12\,p - 3.$

$C = 4\,p^4 + 8\,p^3 + 12\,p^2 + 8\,p + 1.$

$D = 4\,p^4 + 16\,p^3 + 12\,p^2 + 4\,p + 1.$

These results are general and hold good, in the case of the first problem, for all values of p greater than 2.

Taking $p = 3$, we find $A = 864571$; $B = 177$; $C = 673$; $D = 877$;

and these expressions (31), (32) and (33) give for the roots of our numbers, in the case of the first problem, as follows:

$$\frac{m}{a}=\frac{12707211238697}{11011044931800};\ \frac{m}{b}=\frac{18687075351025}{11011044931800};\ \frac{m}{c}=\frac{23171973435271}{11011044931800}$$

In our second problem, the denominator of the expression for m must be negative, and this will be the case for $p=1$, for then we find $A=-389$, $B=-23$, $C=33$, and $D=37$; so that (31), (32) and (33) become

$$\frac{m}{a}=\frac{151321}{7863240};\ \frac{m}{b}=\frac{756605}{7863240};\ \frac{m}{c}=\frac{1059247}{7863240}.$$

If we take $p=-2$, our results will satisfy this second problem, also $p=-3$ will give answers. It will also be satisfied for $p=2$.

Remark.—On page 494, of Stoddard and Henkle's University Algebra, New York, Edition of 1861, our *first* problem is given. The answer is given by x^2, $25\,x^2$ and $49\,x^2$, in very large numbers, consisting of more than three times as many places of figures as in my numbers. The value of the root of the first number is there given

$$x=\frac{2340814414884742932783918468592674193422 5281}{20177642715140781960429281969996251353230160}.$$

It is stated that these numbers were furnished by Prof. Daniel Kirkwood, and that he believed they were the smallest numbers which could be found. His method of solving the problem is not given.

I do not recollect ever having seen a solution of this *first* problem. In an Elementary Treatise on Algebra by John D. Williams, Boston, 1840, on page 413, he gives the solution of our *second* problem. He also proceeds in his solution by assuming x^2, $25\,x^2$ and $49\,x^2$ for the numbers, and obtains for his results the same numbers which I have given.

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SOLUTION OF A PROBLEM.

BY PROF. O ROOT, HAMILTON COLLEGE, CLINTON, N. Y.

Problem.—"From a point in the circumference of a circular field a projectile is thrown at random with a given velocity, which is such that the diameter of the field is equal to the greatest range of the projectile; find the chance of its falling within the field."